February 9, 2008

K-THEORY OF NONCOMMUTATIVE LATTICES

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Abstract

Noncommutative lattices have been recently used as finite topological approximations in quantum physical models. As a first step in the construction of bundles and characteristic classes over such noncommutative spaces, we shall study their K-theory. We shall do it algebraically, by studying the algebraic K-theory of the associated algebras of 'continuous functions' which turn out to be noncommutative approximately finite dimensional (AF) C^*-algebra . We also work out several examples.

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1 Introduction

Topological lattices, namely finite topological spaces, were introduced in [1] (with the name posets or partially ordered sets) as finite topological approximations to 'continuum' spaces. Their ability to capture some of the topological information of the space they approximate has been the motivation for their use in quantum theories [2, 3, 4]. The idea is to construct alternative lattice theories which are able to describe also some topologically non trivial configurations. An example of a promising result in this direction is the construction of θ-states for particles on the poset approximations to a circle. In a suitable limit these states give the usual θ-states for particles on the circle [2, 3]. Also, there is some work on gauge field theories [4].

In [3] it was observed that a poset P is truly a noncommutative space (in fact a noncommutative lattice) since it can be described as the structure space (space of irreducible representations) of a noncommutative C^*-algebra $\mathcal{A}$. Therefore, such an algebra is the analogue of the commutative algebra of complex valued continuous functions defined on any Hausdorff topological space. The algebra $\mathcal{A}$ can be thought of as an algebra of operator valued functions over P. The use of these algebras leads to noncommutative geometry [5] as the natural tool to construct "geometric" structures on noncommutative lattices. It turns out that the algebras which are relevant are approximately finite dimensional (AF) postliminal algebras. There are in general several algebras which are associated with the same space P and several ways of constructing any such an algebra [6]. In this paper we will use a diagrammatic method due to Bratteli [7]. Although this method gives only one $\mathcal{A}$ among all possibilities, it will be enough for our purposes.

In this paper, as a first step in the construction of bundles and characteristi classes over noncommutative lattices, we shall study the K-theory of such noncommutative lattices. We shall do it from the algebraic point of view, by studying the algebraic K-theory of the associated algebras. Algebraically, the analogue of vector bundles over a noncommutative lattice P are projective modules of finite type over the corresponding algebra $\mathcal{A}$. The K-theory group of $\mathcal{A}$ classifies (stable) equivalence classes of such projective modules. For the class of algebras we are interested in, Bratteli diagrams allow to explicitly construct these groups.

The paper is organized as follows. In Section 2 we shall recall the construction of topological lattices as approximation spaces of 'continuum' topological spaces. Section 3 is devoted to the description of the noncommutative algebras associated with the noncommutative lattices, and the description of the Bratteli diagrams. Several examples are worked out. Section 4 treats the notion of projective modules and the K-theory. Again, several examples are presented.

2 Topological Lattices

We shall briefly recall how to construct a topological lattice from any covering of a 'continuum' topological space M while referring to [1, 3, 6] for details. The idea is simply to identify any two points of M which cannot be separated or distinguished by the sets in the covering. The resulting space consists of a finite (or in general a countable) number of points with a nontrivial topology which maintain some of the topological information of the starting space M [1].

If we are given a covering $\mathcal{U} = \{O_\lambda\}$ of M which is a topology for the latter, any two points x and y in M, will be declared to be equivalent, $x \sim y$, if every set O_λ containing either point x or y contains the other too,

$$x \sim y \quad \text{if and only if} \quad x \in O_\lambda \Leftrightarrow y \in O_\lambda \quad \forall \;\; O_\lambda \; . \tag{2.1}$$

Then, we replace M by $P(M) =: M/\sim$ and endow this space with the quotient topology. When M is compact, the number of sets O_λ in $\mathcal{U}$ can be taken to be finite so that $P(M)$ is an approximation to M by a finite number of points. When M is not compact we take it locally compact so that each point has a neighborhood intersected by only finitely many O_λ and $P(M)$ is a "finitary" (countable) approximation to M [1].

In general, the space $P(M)$ is neither Hausdorff (one cannot separate completely any two points) nor T_1 (not all points are closed). However, it is always a a T_0 space [1]. This means that given any two points, there exists at least an open set which contains only one of the points and not the other.

The space $P(M)$ is made a *partially ordered set* (or a *poset*) with a partial order $\preceq$ given by

$$x \preceq y \quad \text{if every open set containing } y \text{ contains also } x \; .$$

The smallest open set O_x containing a point $x \in P(M)$ consists of all y's preceding x: $O_x = \{y \in P(M) \;\; : \;\; y \preceq x\}$. The open sets $\{O_x \; , x \in P(M)\}$ are a basis for the topology of $P(M)$. For spaces with at most a countable number of points, a topology is equivalent to a partial order. Any poset can be represented pictorially by a *Hasse diagram* constructed by arranging its points at different levels and connecting them using the following rules:

1) if $x \prec y$, then x is at a lower level than y;

2) if $x \prec y$ and there is no z such that $x \prec z \prec y$, then x is at the level immediately below y and these two points are connected by a line called a link.

Fig. 1 shows the Hasse diagram for the poset $P_4(S^1)$, a four point approximation to S^1. A basis of open sets for the topology is given by

$$\{x_1\} \; , \quad \{x_2\} \; , \quad \{x_1, x_2, x_3\} \; , \quad \{x_1, x_2, x_4\} \; . \tag{2.2}$$

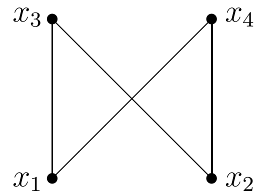


Figure 1: The Hasse diagram for $P_4(S^1)$.

Fig. 2 shows the Hasse diagram for the poset $P_6(S^2)$, a six points approximation to S^2. A basis of open sets for the topology is given by

$$\begin{aligned} &\{x_1\}\ ,\quad \{x_2\}\ ,\quad \{x_1,x_2,x_3\}\ ,\quad \{x_1,x_2,x_4\}\ , \\ &\{x_1,x_2,x_3,x_4,x_5\}\ ,\quad \{x_1,x_2,x_3,x_4,x_6\}\ . \end{aligned} \tag{2.3}$$

As alluded to before, posets maintain some of the topological information of the spaces they approximate. This is shown, for instance, by their ability to reproduce homotopy groups. For example, $\pi_1(P_N(S^1)) = \mathbb{Z}$ whenever $N \geq 4$ [1]. Similarly, $\pi_1(P_6(S^2)) = \{0\}$ and $\pi_2(P_6(S^2)) = \mathbb{Z}$. It is this fact that makes posets relevant for quantum physical models such as θ-states [2, 3]. Finally, we mention that the topological space being approximated can be recovered by considering a sequence of finer and finer coverings, the appropriated framework being that of projective systems of topological spaces. We refer to [1, 8] for details.

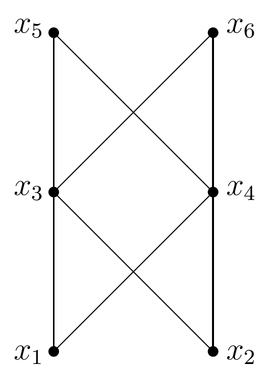


Figure 2: The Hasse diagram for the poset $P_6(S^2)$.

3 Noncommutative Lattices

Associated with any poset P there is a noncommutative C^*-algebra $\mathcal{A}$ of operator valued continuous functions on P. The latter can be identified with the space $\widehat{\mathcal{A}} = Prim\mathcal{A}$ of primitive ideals of $\mathcal{A}$, an ideal of $\mathcal{A}$ being called primitive if it is the kernel of an irreducible representation of $\mathcal{A}$. On $\widehat{\mathcal{A}}$ there is a partial order defined by inclusion of ideals: given any two ideals $I_1, I_2 \in \widehat{\mathcal{A}}$, $I_1 \preceq I_2$, if and only if $I_1 \subseteq I_2$. The space P being at most countable, the resulting topology coincides with the Jacobson topology. Thus any poset is a noncommutative space (in fact a *noncommutative lattice*) [5].

It turns out that the algebras one is working with are particularly simple. They are approximately finite dimensional (AF) postliminal algebras. An AF algebra can be approximated in norm by direct sums of finite dimensional matrix algebras. Postliminal algebras have the remarkable property that their irreducible (unitary) representations are completely characterized by their kernels [9, 10].

In the rest of this section we shall briefly describe a diagrammatic representation of AF algebras due to Bratteli [7] which is very useful for the study of the K-theory of the associated posets. We refer to [6] for a detailed treatment.

3.1 The Algebra of a Poset and Bratteli Diagrams

A C^*-algebra $\mathcal{A}$ is said to be *approximately finite dimensional* (AF) if there exists an increasing sequence

$$\mathcal{A}_0 \stackrel{I_0}{\hookrightarrow} \mathcal{A}_1 \stackrel{I_1}{\hookrightarrow} \mathcal{A}_2 \stackrel{I_2}{\hookrightarrow} \cdots \stackrel{I_{n-1}}{\hookrightarrow} \mathcal{A}_n \stackrel{I_n}{\hookrightarrow} \cdots \tag{3.1}$$

of finite dimensional C^*-subalgebras $\mathcal{A}_n$ of $\mathcal{A}$, with injective *-homomorphisms I_n, such that $\mathcal{A}$ is the norm closure of $\bigcup_n \mathcal{A}_n$. Here the maps I_n are injective *-homomorphisms. Elements of $\mathcal{A}$ are then coherent sequences of the form,

$$a = (a_n)_{n\in\mathbf{N}} \ , \ a_n \in \mathcal{A}_n \ \text{ s.t. } \ \exists N_0 \ , \ a_{n+1} = I_n(a_n) \ , \forall \ n > N_0. \tag{3.2}$$

The norm of any such element is given by

$$||(a_n)_{n\in\mathbf{N}}|| = \lim_{n\to\infty} ||a_n||_{\mathcal{A}_n} \ . \tag{3.3}$$

Since the maps I_n are injective, the expression (3.3) gives directly a true norm and not simply a seminorm and there is no need to quotient out the zero norm elements [11].

There is a very useful diagrammatic representation of the AF algebra (3.1) due to Bratteli [7]. Each subalgebra $\mathcal{A}_n$ is a direct sum of matrix algebras

$$\mathcal{A}_n = \bigoplus_{k=1}^{n_n} M^{(n)}(d_k, \mathbb{C}) \ , \tag{3.4}$$

with $M^{(n)}(d_k, \mathbb{C})$ the algebra of $d_k \times d_k$ matrices with complex coefficients. Given any two such algebras $\mathcal{A}_1 = \bigoplus_{j=1}^{n_1} \mathbb{M}(d_j^{(1)}, \mathbb{C})$ and $\mathcal{A}_2 = \bigoplus_{k=1}^{n_2} \mathbb{M}(d_k^{(2)}, \mathbb{C})$ with an embedding $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$, one can always choose suitable bases in $\mathcal{A}_1$ and $\mathcal{A}_2$ in such a manner to identify $\mathcal{A}_1$ with a subalgebra of $\mathcal{A}_2$ of the following form

$$\mathcal{A}_1 \simeq \bigoplus_{k=1}^{n_2} \left(\bigoplus_{j=1}^{n_1} N_{kj} \mathbb{M}(d_j^{(1)}, \mathbb{C}) \right) . \tag{3.5}$$

Here, with p, q any two nonnegative integers, the symbol $p\mathbb{M}(q, \mathbb{C})$ stands for $\mathbb{M}(q, \mathbb{C}) \otimes \mathbb{C}\mathbb{I}_p$. The nonnegative integers N_{kj} satisfies the condition $\sum_{j=1}^{n_1} N_{kj} d_j^{(1)} = d_k^{(2)}$. One says that the algebra $\mathbb{M}(d_j^{(1)}, \mathbb{C})$ is *partially embedded* in $\mathbb{M}(d_k^{(2)}, \mathbb{C})$ with *multiplicity* N_{kj}. One represents the embedding $\mathcal{A}_1 \hookrightarrow \mathcal{A}_2$ by means of a *diagram* (the Bratteli diagram), which is constructed out of the dimensions d_j $(j = 1, \ldots, n_1)$ and d_k $(k = 1, \ldots, n_2)$ of the diagonal blocks of the two algebras and the numbers N_{kj} that describe the embedding. To construct the diagram one draws two horizontal rows of vertices, the top (bottom) one representing $\mathcal{A}_1$ ($\mathcal{A}_2$) and consisting of n_1 (n_2) vertices labeled by $d_1, \ldots, d_{n_1}$ $(d_1, \ldots, d_{n_2})$. Then for each $j = 1, \ldots, n_1$ and $k = 1, \ldots, n_2$, draw N_{kj} edges between d_j and d_k. One writes $d_j^{(K)} \searrow^p d_k^{(K+1)}$ to denote the fact that $M^{(K)}(d_j^{(K)}, \mathbb{C})$ is embedded in $M^{(K+1)}(d_k^{(K+1)}, \mathbb{C})$ with multiplicity p. For any AF algebra $\mathcal{A}$ one repeats the procedure for each level so obtaining a semi-infinite diagram denoted by $\mathcal{D}(\mathcal{A})$ which completely defines $\mathcal{A}$ up to isomorphisms.

Example 3.1

Consider the subalgebra $\mathcal{A}$ of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on an infinite dimensional (separable) Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, given in the following manner. Let $\mathcal{P}_j$ be the projection operators on $\mathcal{H}_j$, $j = 1, 2$ and $\mathcal{K}(\mathcal{H})$ be the algebra of compact operators on $\mathcal{H}$. The algebra is then

$$\mathcal{A}(\vee) = \mathbb{C}\mathcal{P}_1 + \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathcal{P}_2 \ . \tag{3.6}$$

The use of the symbol $\mathcal{A}(\bigvee)$ will be clear later. This C^*-algebra can be obtained as the direct limit of the following sequence of finite dimensional algebras:

$$\begin{aligned}
\mathcal{A}_0 &= M(1, \mathbb{C}) \\
\mathcal{A}_1 &= M(1, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
\mathcal{A}_2 &= M(1, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
\mathcal{A}_3 &= M(1, \mathbb{C}) \oplus M(4, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
&\vdots \\
\mathcal{A}_n &= M(1, \mathbb{C}) \oplus M(2n-2, \mathbb{C}) \oplus M(1, \mathbb{C}) \\
&\vdots
\end{aligned} \tag{3.7}$$

with $\mathcal{A}_n$ embedded in $\mathcal{A}_{n+1}$ as $M(1,\mathbb{C})\oplus(M(1,\mathbb{C})\oplus M(2n-2,\mathbb{C})\oplus M(1,\mathbb{C}))\oplus M(1,\mathbb{C})$,

$$a_n = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & m_{(2n-2)\times(2n-2)} & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & m_{(2n-2)\times(2n-2)} & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} . \tag{3.8}$$

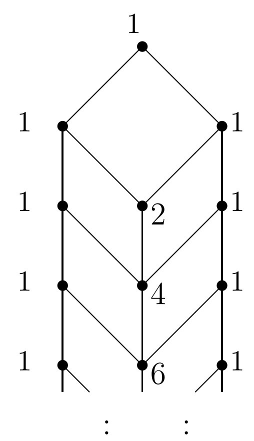


Figure 3: The Bratteli diagram of the algebra $\mathcal{A}(\bigvee)$.

The corresponding Bratteli diagram is shown in Fig. 3.

Out of the Bratteli diagram $\mathcal{D}(\mathcal{A})$ of an AF algebra $\mathcal{A}$ one can also identify the ideals of $\mathcal{A}$ and decide which ones are primitive. As we have mentioned before, the topology is given by constructing a poset whose partial order is provided by inclusion of ideal. In [12] it is proved that the (norm closed) ideals $\{\mathcal{I}\}$ of $\mathcal{A}$ are all and only the (norm closure of) sums of the form

$$\mathcal{I} = \bigcup_{n=1}^{\infty} \oplus_{k,(n,k)\in\Lambda} \ \mathbb{M}^{(n)}(d_k,\mathbb{C}) \ , \tag{3.9}$$

with the subset $\Lambda \equiv \Lambda_{\mathcal{I}}$ of the Bratteli diagram satisfying the following two properties:

i) if $M^{(n)}(d_k,\mathbb{C}) \in \Lambda$ and $M^{(n)}(d_k,\mathbb{C}) \searrow M^{(n+1)}(d_j,\mathbb{C})$ then $M^{(n+1)}(d_j,\mathbb{C})$ belongs to Λ;

ii) if all factors $M^{(n+1)}(d_j,\mathbb{C})$, $j=\{1,2,\cdots,N_{n+1}\}$, in which $M^{(n)}(d_k,\mathbb{C})$ is partially embedded belong to Λ, then $M^{(n)}(d_k,\mathbb{C})$ belongs to Λ.

Furthermore, the ideal (3.9) is primitive if and only if the associated subdiagram $\Lambda_{\mathcal{I}}$ satisfies the additional property:

iii) $\forall\ n$ there exists an $M^{(m)}(d_j, \mathbb{C})$, with $m > n$, not belonging to Λ such that, for all $k \in \{1, 2, \cdots, N_n\}$ with $M^{(n)}(d_k, \mathbb{C})$ not in Λ, one can find a sequence $M^{(n)}(d_k, \mathbb{C}) \searrow M^{(n+1)}(d_\alpha, \mathbb{C}) \searrow M^{(n+2)}(d_\beta, \mathbb{C}) \searrow \cdots \searrow M^{(m)}(d_j, \mathbb{C})$.

The whole $\mathcal{A}$ is an ideal which, by definition, is not primitive since the trivial representation $\mathcal{A} \to 0$ is not irreducible. Furthermore, the ideal $\{0\} \subset \mathcal{A}$ is primitive if and only if $\mathcal{A}$ has one irreducible faithful representation. This can be understood from the Bratteli diagram in the following way. The set $\{0\}$ is not a subdiagram of $\mathcal{D}(\mathcal{A})$, being represented by the element 0 of the matrix algebra of each finite level, so that there is at least one element $a \in \mathcal{A}$ not belonging to the ideal $\{0\}$ at any level. Thus to check if $\{0\}$ is primitive, i.e. to check property (iii) above, we have to see whether we can connect all the points at a level n to a *single* point at a level $m > n$. For example this is the case for the diagram of Fig. 3. Later, we shall construct examples in which $\{0\}$ is not a primitive ideal.

Thus one can understand the topological properties of $Prim(\mathcal{A})$ at once from the Bratteli diagram $\mathcal{D}(\mathcal{A})$. This is particularly simple if the algebra admits only a finite number of nonequivalent irreducible representations. In this case $Prim(\mathcal{A})$ is a T_0-topological space with only a finite number of points, hence a finite poset P. To reconstruct the latter we just need to draw the Bratteli diagram $\mathcal{D}(\mathcal{A})$ and find the subdiagrams that, according to properties (i,ii,iii), correspond to primitive ideals. Then P has as many points as the number of primitive ideals and the partial order relation in P is simply given by the inclusion relations that exist among the primitive ideals.

Example 3.2
Again, consider the diagram of Fig. 3. The corresponding AF algebra $\mathcal{A}$ in (3.6) contains only three nontrivial ideals, whose diagrams are represented in Fig. 4(a,b,c). In this pictures the points belonging to the ideals are marked with a "$\clubsuit$". It is not difficult to check that only $\mathcal{I}_1$ and $\mathcal{I}_2$ are primitive ideals, since $\mathcal{I}_3$ does not satisfy property (iii) above. Clearly, $\{0\}$ is primitive and belongs to both $\mathcal{I}_1$ and $\mathcal{I}_2$ so that $Prim(\mathcal{A})$ is the $\bigvee$ poset of Fig. 5.

3.2 From Posets to Bratteli Diagrams

Under some rather mild hypotheses which are always verified in the cases of posets, it is possible to reverse the construction of previous section and thus construct an AF algebra

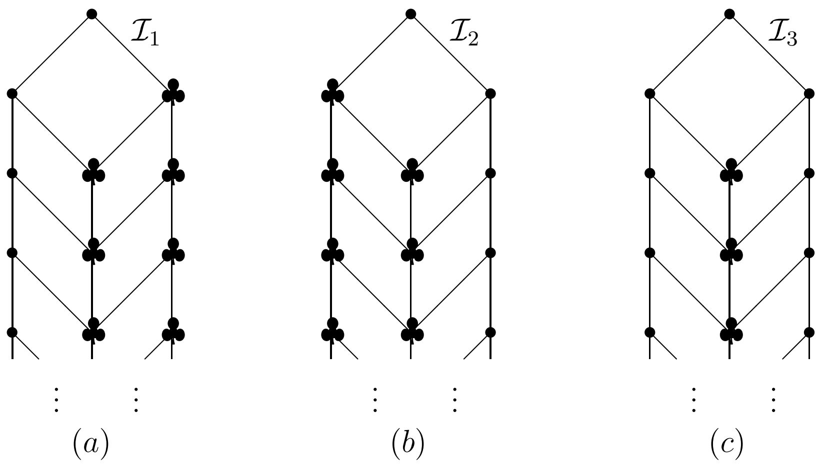


Figure 4: The three ideals of the algebra $\mathcal{A}(\bigvee)$.

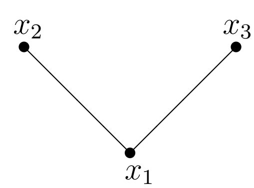


Figure 5: The poset $\bigvee$ as the primitive spectrum of the algebra $\mathcal{A}(\bigvee)$.

starting from a poset. Such a reconstruction rests on another result by Bratteli [12], which specifies the class of topological spaces which are the primitive ideal spaces of AF algebras. Here, by using the techniques of [12] we shall explain how to explicitly find an AF algebra $\mathcal{A}$ (or rather its Bratteli diagram $\mathcal{D}(\mathcal{A})$) whose primitive ideal space is a given finite poset P. First we will give the general construction and then discuss several examples.

Let $\{K_1, K_2, K_3, \ldots\}$ be the collection of all closed sets in the poset P, with $K_1 = P$. To construct the n-th level of the Bratteli diagram $\mathcal{D}(\mathcal{A})$, we consider the subcollection $\mathcal{K}_n = \{K_1, K_2, \ldots, K_n\}$ and denote with $\mathcal{K}'_n$ the smallest collection of (closed) sets in P containing $\mathcal{K}_n$ which is closed under union and intersection. The collection $\mathcal{K}_n$ determines a partition of the space P by taking intersections and complements of the sets $K_j \in \mathcal{K}_n$ $(j = 1, \ldots, n)$. We denote with $Y(n,1)$, $Y(n,2)$, $\ldots$, $Y(n,k_n)$ the minimal sets of such partition. Also, we write $F(n,j)$ for the smallest set in the subcollection $\mathcal{K}'_n$ which contains $Y(n,j)$. Then, the diagram $\mathcal{D}(\mathcal{A})$ is constructed as follows:

1. the n-th level of $\mathcal{D}(\mathcal{A})$ has k_n points, one for each set $Y(n,k), k = 1, \cdots, k_n$;

2. at the level n of the diagram, the point which corresponds to $Y(n,i)$ is linked to the point at the level $n+1$ corresponding to $Y(n+1,j)$ if and only if $Y(n,i) \cap F(n+1,j) \neq \emptyset$. In this case, the multiplicity of the embedding is always 1.

Example 3.3
To illustrate this construction, let us consider again the $\bigvee$ poset of Fig. 5, $P = \{x_1, x_2, x_3\}$. This topological space contains four closed sets:

$$K_1 = \{x_1, x_2, x_3\} \ , K_2 = \{x_2\} \ , K_3 = \{x_3\} \ , K_4 = \{x_2, x_3\} = K_2 \cup K_3 \ . \tag{3.10}$$

Thus it is not difficult to check that:

$$\begin{array}{llll}
\mathcal{K}_1 = \{K_1\} & \mathcal{K}'_1 = \{K_1\} & Y(1,1) = \{x_1, x_2, x_3\} & F(1,1) = K_1 \\
\mathcal{K}_2 = \{K_1, K_2\} & \mathcal{K}'_2 = \{K_1, K_2\} & Y(2,1) = \{x_2\} & F(2,1) = K_2 \\
 & & Y(2,2) = \{x_1, x_3\} & F(2,2) = K_1 \\
\mathcal{K}_3 = \{K_1, K_2, K_3\} & \mathcal{K}'_3 = \{K_1, K_2, K_3, K_4\} & Y(3,1) = \{x_2\} & F(3,1) = K_2 \\
 & & Y(3,2) = \{x_1\} & F(3,2) = K_1 \\
 & & Y(3,3) = \{x_3\} & F(3,3) = K_3 \\
\mathcal{K}_4 = \{K_1, K_2, K_3, K_4\} & \mathcal{K}'_4 = \{K_1, K_2, K_3, K_4\} & Y(4,1) = \{x_2\} & F(4,1) = K_2 \\
 & & Y(4,2) = \{x_1\} & F(4,2) = K_1 \\
 & & Y(4,3) = \{x_3\} & F(4,2) = K_3 \\
 & \vdots & &
\end{array} \tag{3.11}$$

Since P has only a finite number of points and hence a finite number of closed sets, the partition of P repeats itself after a certain level ($n = 3$ in this case). The corresponding diagram, obtained through rules (1) and (2) above is readily found to coincide with the one in Fig. 3. As we have said previously, such a diagram corresponds to the AF algebra $\mathcal{A}(\bigvee) = \mathbb{C}\mathcal{P}_1 + \mathcal{K}(\mathcal{H}) + \mathbb{C}\mathcal{P}_2$.

It is a general fact that a Bratteli diagram describing any (finite) poset 'stabilizes' after a certain level n_0, namely it repeats itself . From that level, both the number of points, as well as the embeddings from one level to the next one, do not change. As we shall mention later, for the construction of the K-theory groups, we only need the stable part of the Bratteli diagram and we shall construct this part only in the remaining examples.

Now, the Bratteli diagram stabilizes at the level n_0 if the family $\mathcal{K}_{n_0}$ of closed sets we choose is such that it determines a partition of the poset which distinguishes each point of the poset itself. In particular, this is the case if we choose n_0 in such a manner that $\mathcal{K}_{n_0}$ contains all closed set. Then, each $Y(n_0, j)$ will contain a single point of the poset and $F(n_0+1, j)$ will be the smallest closet set containing $Y(n_0, j)$.

Example 3.4

The poset $|\!/\!|$ of Fig. 6. Here $n_0 = 4$ and the stable partition is given by

$$\begin{array}{ll} Y(n_0,1) = \{x_3\} & F(n_0+1,1) = \{x_3\} \\ Y(n_0,2) = \{x_1\} & F(n_0+1,2) = \{x_1, x_3, x_4\} \\ Y(n_0,3) = \{x_2\} & F(n_0+1,3) = \{x_2, x_4\} \\ Y(n_0,4) = \{x_4\} & F(n_0+1,4) = \{x_4\}\ . \end{array} \tag{3.12}$$

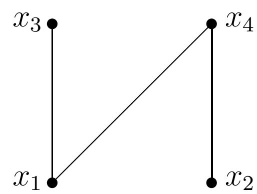


Figure 6: The poset $|\!/\!|$.

The corresponding Bratteli diagram is shown in Fig. 7. The set $\{0\}$ is not an ideal. The algebra limit $\mathcal{A}(|\!/\!|)$ turns out to be a subalgebra of bounded operators on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ with $\mathcal{H}_i,\ i = 1, 2, 3$ infinite dimensional Hilbert spaces [6],

$$\mathcal{A}(|\!/\!|) = \mathbb{C}\mathcal{P}(\mathcal{H}_1) \oplus \mathbb{C}\mathcal{P}(\mathcal{H}_2 \oplus \mathcal{H}_3) \oplus \mathcal{K}(\mathcal{H}_2 \oplus \mathcal{H}_2) \oplus \mathcal{K}(\mathcal{H}_3)\ . \tag{3.13}$$

Here, $\mathcal{K}$ denotes the set of compact operators and $\mathcal{P}$ orthogonal projection.

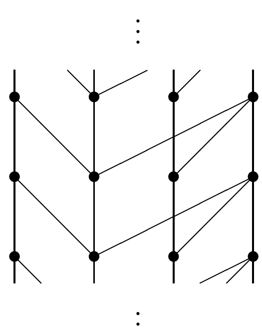


Figure 7: The stable part of the Bratteli diagram of the poset $|\!/\!|$.

Example 3.5
The poset $P_4(S^1)$ for the one-dimensional sphere in Fig. 1. Here $n_0 = 4$ and the stable partition is given by

$$\begin{array}{ll} Y(n_0,1) = \{x_3\} & F(n_0+1,1) = \{x_3\} \\ Y(n_0,2) = \{x_1\} & F(n_0+1,2) = \{x_1,x_3,x_4\} \\ Y(n_0,3) = \{x_2\} & F(n_0+1,3) = \{x_2,x_3,x_4\} \\ Y(n_0,4) = \{x_4\} & F(n_0+1,4) = \{x_4\}\ . \end{array} \tag{3.14}$$

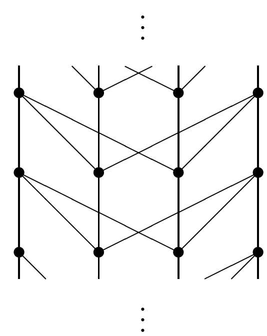


Figure 8: The stable part of the Bratteli diagram for the circle poset $P_4(S^1)$.

The corresponding Bratteli diagram is shown in Fig. 8. The set $\{0\}$ is not an ideal. The algebra limit $\mathcal{A}(P_4(S^1))$ turns out to be a subalgebra of bounded operators on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_4$, with $\mathcal{H}_i,\ i = 1,\ldots,4$ infinite dimensional Hilbert spaces [6],

$$\mathcal{A}(P_4(S^1)) = \mathbb{C}\mathcal{P}(\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathbb{C}\mathcal{P}(\mathcal{H}_2 \oplus \mathcal{H}_3) \oplus \mathcal{K}(\mathcal{H}_1 \oplus \mathcal{H}_2) \oplus \mathcal{K}(\mathcal{H}_3 \oplus \mathcal{H}_4)\ . \tag{3.15}$$

Here $\mathcal{K}$ denotes the set of compact operators and $\mathcal{P}$ orthogonal projection.

Example 3.6
The poset $P_6(S^2)$ for the two-dimensional sphere in Fig. 2. Here $n_0 = 6$ and the stable partition is given by

$$\begin{array}{ll} Y(n_0,1) = \{x_5\} & F(n_0+1,1) = \{x_5\} \\ Y(n_0,2) = \{x_3\} & F(n_0+1,2) = \{x_3,x_5,x_6\} \\ Y(n_0,3) = \{x_1\} & F(n_0+1,3) = \{x_1,x_3,x_4,x_5,x_6,\} \\ Y(n_0,4) = \{x_2\} & F(n_0+1,4) = \{x_2,x_3,x_4,x_5,x_6\} \\ Y(n_0,5) = \{x_4\} & F(n_0+1,5) = \{x_4,x_5,x_6\} \\ Y(n_0,6) = \{x_6\} & F(n_0+1,6) = \{x_6\}\ . \end{array} \tag{3.16}$$

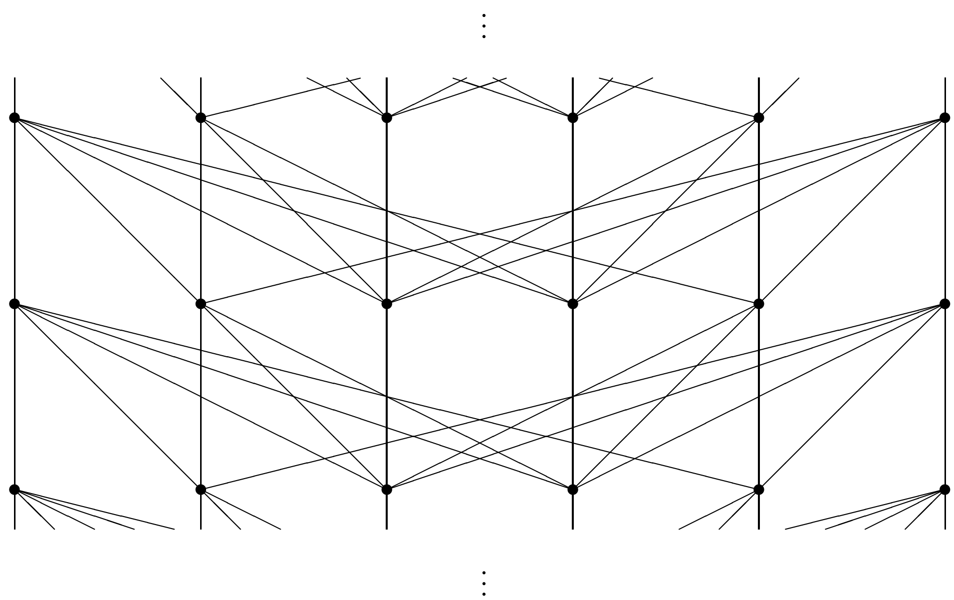


Figure 9: The stable part of the Bratteli diagram for the sphere poset $P_6(S^2)$.

The corresponding Bratteli diagram is shown in Fig. 9. The set $\{0\}$ is not an ideal. The algebra limit $\mathcal{A}(P_6(S^2))$ turns out to be a subalgebra of bounded operators on the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_8$ with $\mathcal{H}_i,\ i = 1, \ldots, 8$ infinite dimensional Hilbert spaces [6],

$$\begin{aligned}\mathcal{A}(P_6(S^2)) = {} & \mathbb{C}\mathcal{P}\left(\mathcal{H}_5 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{H}_8 \otimes (\mathcal{H}_2 \oplus \mathcal{H}_3)\right) \\ & \oplus \mathbb{C}\mathcal{P}\left(\mathcal{H}_6 \otimes (\mathcal{H}_1 \oplus \mathcal{H}_4) \oplus \mathcal{H}_7 \otimes (\mathcal{H}_2 \oplus \mathcal{H}_3)\right) \\ \oplus & \left[\mathcal{K}(\mathcal{H}_5 \oplus \mathcal{H}_6) \otimes \mathbb{C}\mathcal{P}(\mathcal{H}_1 \oplus \mathcal{H}_4)\right] \oplus \left[\mathcal{K}(\mathcal{H}_7 \oplus \mathcal{H}_8) \otimes \mathbb{C}\mathcal{P}(\mathcal{H}_2 \oplus \mathcal{H}_3)\right] \\ \oplus & \mathcal{K}\left[\mathcal{H}_1 \otimes (\mathcal{H}_5 \oplus \mathcal{H}_6) \oplus \mathcal{H}_2 \otimes (\mathcal{H}_7 \oplus \mathcal{H}_8)\right] \oplus \mathcal{K}\left[\mathcal{H}_3 \otimes (\mathcal{H}_7 \oplus \mathcal{H}_8) \oplus \mathcal{H}_4 \otimes (\mathcal{H}_5 \oplus \mathcal{H}_6)\right]\end{aligned} \tag{3.17}$$

Here $\mathcal{K}$ denotes the set of compact operators and $\mathcal{P}$ orthogonal projection.

4 Projective modules of finite type and K-theory

Given an algebra $\mathcal{A}$ playing the role of the algebra of continuous functions on some non-commutative space, the analogue of vector bundles is provided by the notion of *projective*

module of finite type (or *finite projective module*) over $\mathcal{A}$. Indeed, by the Serre-Swan theorem [13], locally trivial, finite-dimensional complex vector bundles over a compact Hausdorff space M are in one to one correspondence with finite projective modules over the algebra $\mathcal{A} = \mathcal{C}(M)$. To the vector bundle E one associates the $\mathcal{C}(M)$-module $\mathcal{E} = \Gamma(M, E)$ of continuous sections of E. Conversely, if $\mathcal{E}$ is a finite projective modules over $\mathcal{C}(M)$, the fiber E_m of the associated bundle E over the point $m \in M$ is

$$E_m = \mathcal{E}/\mathcal{E}\mathcal{I}_m \ , \tag{4.1}$$

where the ideal $\mathcal{I}_m \subset \mathcal{C}(M)$, corresponding to the point $m \in M$, is given by [5, 14]

$$\mathcal{I}_m = ker\{\chi_m : \mathcal{C}(M) \to \mathbb{C} \ ; \ \chi_m(f) = f(m)\} = \{f \in \mathcal{C}(M) \mid f(m) = 0\} \ . \tag{4.2}$$

As we shall see, isomorphism and stable isomorphism have a meaning in the context of finite projective modules over a C^*-algebra $\mathcal{A}$ and the group $K_0(\mathcal{A})$ will be the group of (stable) isomorphism classes of finite projective (right) modules over $\mathcal{A}$.

Given any finite projective right module $\mathcal{E}$ over $\mathcal{A}$, there exists an integer N together with an idempotent $p \in \mathbb{M}(N, \mathcal{A})$ ($N \times N$ matrix with entries in $\mathcal{A}$ and $p^2 = p$), and an isomorphism of $\mathcal{E}$ with the right $\mathcal{A}$-module [5]

$$p\mathcal{A}^N = \{\xi = (\xi_1, \dots, \xi_N) \ ; \ \xi_i \in \mathcal{A} \ , \ p\xi = \xi\} \ . \tag{4.3}$$

In fact, we shall restrict to the class of *hermitian* modules which correspond to projectors, namely idempotents p obeying the additional condition $p^* = p$, the operation * being the composition of the * operation in the algebra $\mathcal{A}$ with usual matrix transposition.

We shall now give few fundamentals of the K-theory of C^*-algebras having in mind mainly AF algebras [11]. Two projectors $p, q \in \mathbb{M}(N, \mathcal{A})$ are equivalent if there exists a matrix $u \in \mathbb{M}(N, \mathcal{A})$ such that $p = u^*u$ and $q = uu^*$. In order to be able to add equivalence classes of projectors, one considers all finite matrix algebras over $\mathcal{A}$ at the same time, by considering $\mathbb{M}(\infty, \mathcal{A})$ which is the non complete *-algebra obtained as inductive limit of finite matrices [1],

$$\begin{aligned} &\mathbb{M}(\infty, \mathcal{A}) = \bigcup_{n=1}^{\infty} \mathbb{M}(n, \mathcal{A}) \ , \\ &\phi : \mathbb{M}(n, \mathcal{A}) \to \mathbb{M}(n+1, \mathcal{A}) \ , \ a \mapsto \phi(a) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \ . \end{aligned} \tag{4.4}$$

Now, two projectors $p, q \in \mathbb{M}(\infty, \mathcal{A})$ are said to be equivalent, $p \sim q$, when there exists an $u \in \mathbb{M}(\infty, \mathcal{A})$ such that $p = u^*u$ and $q = uu^*$. The set $V(\mathcal{A})$ of equivalence classes $[\ \cdot \ ]$ is made into an abelian semigroup by defining an *addition* by

$$[p] + [q] =: [\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}] \ . \tag{4.5}$$

[1]The completion of $\mathbb{M}(\infty, \mathcal{A})$ is $\mathcal{A} \otimes \mathcal{K}$, with $\mathcal{K}$ the algebra of compact operators on l_2. The algebra $\mathcal{A} \otimes \mathcal{K}$ is also called the stabilization of $\mathcal{A}$.

The group $K_0(\mathcal{A})$ is the universal canonical group associated with the semigroup $V(\mathcal{A})$ and may be defined as

$$\begin{aligned} & K_0(\mathcal{A}) =: V(\mathcal{A}) \times V(\mathcal{A})/\sim \ , \quad ([p],[q]) \sim ([p'],[q']) \ , \\ & \Leftrightarrow \quad \text{there exists } [r] \in V(\mathcal{A}) \ \text{ s.t. } \ [p]+[q']+[r] = [p']+[q]+[r] \ . \end{aligned} \tag{4.6}$$

The extra $[r]$ is necessary to get transitivity and make $\sim$ an equivalence relation. This is the reason why one is classifying only stable classes. From definition (4.6), an equivalence class $[([p],[q])] \in K_0(\mathcal{A})$ can also be written as a formal difference $[p]-[q]$.

There is a natural homomorphism

$$\kappa_{\mathcal{A}} : V(\mathcal{A}) \to K_0(\mathcal{A}) \ , \quad \kappa_{\mathcal{A}}([p]) =: [p]-[0] \tag{4.7}$$

This map is injective if and only if the addition in $V(\mathcal{A})$ has cancellations, namely if and only if $[p]+[r]=[q]+[r] \ \Rightarrow [p]=[q]$.

While for a generic $\mathcal{A}$, the semigroup $V(\mathcal{A})$ has no cancellations, for AF algebras this happens to be the case. By defining

$$K_{0+}(\mathcal{A}) =: \kappa_{\mathcal{A}}(V(\mathcal{A})) \ , \tag{4.8}$$

the couple $(K_0(\mathcal{A}), K_{0+}(\mathcal{A}))$ becomes, for an AF algebra $\mathcal{A}$, an *ordered group* with $K_{0+}(\mathcal{A})$ the *positive cone*, namely one has that

$$\begin{aligned} & K_{0+}(\mathcal{A}) \ni 0 \ , \\ & K_{0+}(\mathcal{A}) - K_{0+}(\mathcal{A}) = K_0(\mathcal{A}) \ , \\ & K_{0+}(\mathcal{A}) \cap (-K_{0+}(\mathcal{A})) = 0 \ . \end{aligned} \tag{4.9}$$

For a generic (unital) algebra the last property is not true and the couple $(K_0(\mathcal{A}), K_{0+}(\mathcal{A}))$ is not an ordered group.

There is another K-group, K_1, which is constructed from unitaries or invertibles in $\mathbb{M}(\infty, \mathcal{A})$. It turns out, however, that such a group is trivial for AF algebras, namely $K_1(\mathcal{A}) = 0$ for any AF algebras [11]. We shall not mention it anymore in the rest of the paper.

The construction of the K-theory of AF algebras is made easy by the following results whose proofs are given in [11].

Proposition 4.1 *If $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism of C^*-algebras , then the induced map*

$$\alpha_* : V(\mathcal{A}) \to V(\mathcal{B}) \ , \quad \alpha_*([a_{ij}]) =: [\alpha(a_{ij})] \ , \tag{4.10}$$

is a well defined homomorphism of semigroups. Moreover, from universality, α_* extends to a group homomorphism

$$\alpha_* = K_0(\mathcal{A}) \to K_0(\mathcal{B}) \ . \tag{4.11}$$

Proposition 4.2 *If $\mathcal{A}$ is the inductive limit of a directed system $\{\mathcal{A}_i, \Phi_{ij}\}_I$ of C^*-algebras , then $\{K_0(\mathcal{A}_i), \Phi_{ij*}\}_I$ is a directed system of groups and one can exchange the limits,*

$$K_0(\mathcal{A}) = K_0(\lim_{\rightarrow} \mathcal{A}_i) = \lim_{\rightarrow} K_0(\mathcal{A}_i) \ . \tag{4.12}$$

Moreover, if $\mathcal{A}$ is an AF algebra, then $K_0(\mathcal{A})$ is an ordered group with positive cone given by the limit of a directed system of semigroups

$$K_{0+}(\mathcal{A}) = K_{0+}(\lim_{\rightarrow} \mathcal{A}_i) = \lim_{\rightarrow} K_{0+}(\mathcal{A}_i) \ . \tag{4.13}$$

Proposition 4.3 *With $k_{\mathcal{A}}, k_{\mathcal{B}}$ integer numbers, let $\mathcal{A}$ and $\mathcal{B}$ be the sum of $k_{\mathcal{A}}$ and $k_{\mathcal{B}}$ matrix algebras respectively,*

$$\begin{aligned} \mathcal{A} &= \mathbb{M}(p_1, \mathbb{C}) \oplus \mathbb{M}(p_2, \mathbb{C}) \oplus \cdots \oplus \mathbb{M}(p_{k_{\mathcal{A}}}, \mathbb{C}) \ , \\ \mathcal{B} &= \mathbb{M}(q_1, \mathbb{C}) \oplus \mathbb{M}(q_2, \mathbb{C}) \oplus \cdots \oplus \mathbb{M}(q_{k_{\mathcal{B}}}, \mathbb{C}) \ . \end{aligned} \tag{4.14}$$

Then, any homomorphism $\alpha : \mathcal{A} \rightarrow \mathcal{B}$ can be written as the direct sum of the representations $\alpha_j : \mathcal{A} \rightarrow \mathbb{M}(q_j, \mathbb{C}) \simeq \mathcal{B}(\mathbb{C}^{q_j})$, $j = 1, \cdots, k_{\mathcal{B}}$. If π_{ij} is the unique irreducible representation of $\mathbb{M}(p_i, \mathbb{C})$ in $\mathcal{B}(\mathbb{C}^{q_j})$, then α_j breaks into a direct sum of the π_{ij}. Furthermore, let m_{ij} be the non-negative integer denoting the multiplicity of π_{ij} in this sum. Then the induced homomorphism, $\alpha_* = K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$, is given by the $q_{k_{\mathcal{A}}} \times p_{k_{\mathcal{B}}}$ matrix (m_{ij}).

We end this section by mentioning that K-theory has been proved [15] to be a complete invariant which distinguish among AF algebras if one add to the ordered group $(K_0(\mathcal{A}), K_{0+}(\mathcal{A}))$ the notion of *scale*, the latter being defined for any C^*-algebra $\mathcal{A}$ as

$$\Sigma\mathcal{A} =: \{[p] \ , p \ \text{ a projector in } \mathcal{A}\} \ . \tag{4.15}$$

AF algebras are completely determined, up to isomorphism, by their *scaled ordered* groups, namely by triple (K_0, K_{0+}, Σ). The key is the fact that scale preserving isomorphisms between the ordered groups (K_0, K_{0+}, Σ) of two AF algebras are nothing but K-theoretically induced maps (4.11) of isomorphisms between the AF algebras themselves.

4.1 The K-theory of noncommutative lattices

The starting point to compute the ordered group (K_0, K_{0+}) for a poset is the fact that, for an AF algebra given as in 3.1, the group $(K_0(\mathcal{A}), K_{0+}(\mathcal{A}))$ is obtained by Proposition 4.2 as the inductive limit of the sequence of groups/semigroups

$$K_0(\mathcal{A}_1) \hookrightarrow K_0(\mathcal{A}_2) \hookrightarrow K_0(\mathcal{A}_3) \hookrightarrow \cdots \tag{4.16}$$

$$K_{0+}(\mathcal{A}_1) \hookrightarrow K_{0+}(\mathcal{A}_2) \hookrightarrow K_{0+}(\mathcal{A}_3) \hookrightarrow \cdots \tag{4.17}$$

The inclusions

$$T_n : K_0(\mathcal{A}_n) \hookrightarrow K_0(\mathcal{A}_{n+1}) \ , \quad T_n : K_{0+}(\mathcal{A}_n) \hookrightarrow K_{0+}(\mathcal{A}_{n+1}) \ , \tag{4.18}$$

are easily obtained from the inclusions $\mathcal{A}_n \overset{I_n}{\hookrightarrow} \mathcal{A}_{n+1}$, being indeed the corresponding induced maps as in (4.11). As sets we have that

$$K_0(\mathcal{A}) = \{(k_n)_{n\in\mathbb{N}} \ , k_n \in K_0(\mathcal{A}_n) \mid \exists N_0 \ : \ k_{n+1} = T_n(k_n) \ , \ n > N_0\} \ , \tag{4.19}$$
$$K_{0+}(\mathcal{A}) = \{(k_n)_{n\in\mathbb{N}} \ , k_n \in K_{0+}(\mathcal{A}_n) \mid \exists N_0 \ : \ k_{n+1} = T_n(k_n) \ , \ n > N_0\} \ . \tag{4.20}$$

The structure of (abelian) group/semigroup is inherited pointwise from the addition in the groups/semigroups in the sequences (4.16), (4.17).

Furthermore, for any d, the algebra of matrices $\mathbb{M}(d, \mathbb{C})$ has K-theory given by $(K_0, K_{0+}) = (\mathbb{Z}, \mathbb{Z}_+)$, $\mathbb{Z}$ being the group of integer numbers and $\mathbb{Z}_+$ the semigroups of natural numbers (including 0). Hence, all terms in the sequences (4.16), (4.17), are direct sums of copies of $\mathbb{Z}$ or $\mathbb{Z}_+$.

As mentioned in Section 3.2, the Bratteli diagrams that describe (finite) posets have the property that starting from a certain level n_0 (which is less or equal than the number of closed sets in the poset), the number of points in any diagram, as well as the embeddings from one level to the next one, does not change. This simplifies the calculation of (K_0, K_{0+}) because,

$$K_0(\mathcal{A}_{n_0}) = K_0(\mathcal{A}_{n_0+1}) = K_0(\mathcal{A}_{n_0+2}) = \cdots = \mathbb{Z}^{\oplus k_{n_0}} \ , \tag{4.21}$$
$$K_{0+}(\mathcal{A}_{n_0}) = K_{0+}(\mathcal{A}_{n_0+1}) = K_{0+}(\mathcal{A}_{n_0+2}) = \cdots = \mathbb{Z}_+^{\oplus k_{n_0}} \ , \tag{4.22}$$

where k_{n_0} is the number of points in the Bratteli diagram from the level n_0 on. Furthermore, the integer valued matrices T_n in (4.18) are all equal for $n > n_0$. To find the group $(K_0(\mathcal{A}), K_{0+}(\mathcal{A}))$ one has just to study the limit for $n \to \infty$ of the inclusions

$$T_n : \mathbb{Z}^{\oplus k_{n_0}} \hookrightarrow \mathbb{Z}^{\oplus k_{n_0}} \ , \tag{4.23}$$
$$T_n : \mathbb{Z}_+^{\oplus k_{n_0}} \hookrightarrow \mathbb{Z}_+^{\oplus k_{n_0}} \ . \tag{4.24}$$

We infer from Prop. 4.3 that for AF algebras the maps (4.23), (4.24) are always inclusions. In fact, for (finite) posets, the map (4.23) is always a bijection. As a consequence, for a poset P with k_{n_0} points in the stable part of the corresponding Bratteli diagram, and associated algebra $\mathcal{A}_{k_{n_0}}(P)$, we shall have that

$$K_0(P) = \mathbb{Z}^{\oplus k_{n_0}} \ . \tag{4.25}$$

The map (4.24) will not be in general a bijection.

We shall illustrate the construction of the K-groups with the Penrose Tiling AF algebra. Although this algebra is quite far from being postliminal, since there are infinite

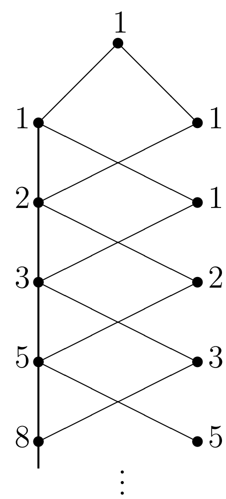


Figure 10: The Bratteli diagram for the algebra of the Penrose tiling.

non equivalent representations with the same kernel (the only primitive ideal), the construction of its K-theory is illuminating. The corresponding Bratteli diagram is shown in Fig. 10. At each level, the algebra is given by [5]

$$\mathcal{A}_n = \mathbb{M}(d_n, \mathbb{C}) \oplus \mathbb{M}(d'_n, \mathbb{C}) \ , \quad n \geq 1 \ , \tag{4.26}$$

with inclusion $\mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$,

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \rightarrow \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix} \ , \quad A \in \mathbb{M}(d_n, \mathbb{C}) \ , \quad B \in \mathbb{M}(d'_n, \mathbb{C}) \ . \tag{4.27}$$

After the second level we have then

$$K_0(\mathcal{A}_n) = \mathbb{Z} \oplus \mathbb{Z} \ , \quad K_{0+}(\mathcal{A}_n) = \mathbb{Z}_+ \oplus \mathbb{Z}_+ \ . \tag{4.28}$$

The inclusion (4.27) gives for the dimensions

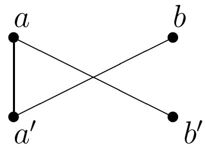
$$\begin{aligned} d_{n+1} &= d_n + d'_n \ , \\ d'_{n+1} &= d_n \ . \end{aligned} \tag{4.29}$$

while the inclusions (4.23), (4.24) are both represented by the integer valued matrix

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ . \tag{4.30}$$

The action of the matrix (4.30) can be represented pictorially as in Fig. 11 where the couple (a, b) (a', b') are both in $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z}_+ \oplus \mathbb{Z}_+$.

Finally, we can construct the K-theory groups.



$$\Rightarrow \begin{cases} a' = a + b \\ b' = a \end{cases}$$

Figure 11: The action of the inclusion T.

1) The group $K_0(\mathcal{A})$ is given by

$$K_0(\mathcal{A}) = \mathbb{Z} \oplus \mathbb{Z} \ . \tag{4.31}$$

This follows immediately from the fact that the matrix T in (4.30) is invertible over the integer, its inverse being

$$T^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \ . \tag{4.32}$$

Now, from the definition of inductive limit we have that,

$$K_0(\mathcal{A}) = \{(k_n)_{n \in \mathbb{N}} \ , k_n \in K_0(\mathcal{A}_n) \mid \exists N_0 \ : \ k_{n+1} = T(k_n) \ , \ n > N_0\}. \tag{4.33}$$

In addition, T being a bijection, for any $k_{n+1} \in K_0(\mathcal{A}_{n+1})$, there exist an unique $k_n \in K_0(\mathcal{A}_n)$ such that $k_{n+1} = T k_n$. Thus, $K_0(\mathcal{A}) = K_0(\mathcal{A}_n) = \mathbb{Z} \oplus \mathbb{Z}$.

2) The group $K_{0+}(\mathcal{A})$ is given by

$$K_{0+}(\mathcal{A}) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \ : \frac{1 + \sqrt{5}}{2} a + b \geq 0\} \ . \tag{4.34}$$

Since T is not invertible over $\mathbb{Z}_+$, $K_{0+}(\mathcal{A}) \neq \mathbb{Z}_+ \oplus \mathbb{Z}_+$. To construct $K_{0+}(\mathcal{A})$, we study the image $T(K_{0+}(\mathcal{A}_n))$ in $K_{0+}(\mathcal{A}_{n+1})$. It is easily found to be

$$\begin{aligned} T(K_{0+}(\mathcal{A}_n)) &= \{(a_{n+1}, b_{n+1}) \in \mathbb{Z}_+ \oplus \mathbb{Z}_+ \ : a_{n+1} \geq b_{n+1}\} \\ &\neq K_{0+}(\mathcal{A}_{n+1}) \ . \end{aligned} \tag{4.35}$$

Now, T being injective, $T(K_{0+}(\mathcal{A}_n)) = T(\mathbb{Z}_+ \oplus \mathbb{Z}_+) \simeq \mathbb{Z}_+ \oplus \mathbb{Z}_+$. The inclusion of $T(K_{0+}(\mathcal{A}_n))$ into $K_{0+}(\mathcal{A}_{n+1})$ is shown in Fig. 12. If we identify the subset $T(K_{0+}(\mathcal{A}_n)) \subset K_{0+}(\mathcal{A}_{n+1})$ with $K_{0+}(\mathcal{A}_n)$, we can think of $T^{-1}(K_{0+}(\mathcal{A}_{n+1}))$ as a subset of $\mathbb{Z} \oplus \mathbb{Z}$ and of $T^{-1}(K_{0+}(\mathcal{A}_n))$ as the standard positive cone $\mathbb{Z}_+ \oplus \mathbb{Z}_+$. The result is shown in Fig. 13. Next iteration, namely $T^{-2}(K_{0+}(\mathcal{A}_n))$ is shown in Fig. 14. From definition (4.20), by going to the limit we shall have $K_{0+}(\mathcal{A}) = \lim_{m \to \infty} T^{-m}(\mathbb{Z}_+ \oplus \mathbb{Z}_+)$ and the limit will be a subset of $\mathbb{Z} \oplus \mathbb{Z}$ since T is invertible only over $\mathbb{Z}$. The limit can be easily found [2]. From the defining relation

[2]We owe the following method to J. Varilly.

$F_{m+1} = F_m + F_{m-1}, m \geq 1$, for the Fibonacci numbers (with $F_0 = 0, F_1 = 1$), it follows that

$$T^{-m} = (-1)^m \begin{bmatrix} F_{m-1} & -F_m \\ -F_m & F_{m+1} \end{bmatrix} . \tag{4.36}$$

Therefore, T^{-m} takes the positive axis $\{(a, 0) : a \geq 0\}$ to an half-line of slope $-F_m/F_{m-1}$, and the positive axis $\{(0, b) : b \geq 0\}$ to an half-line of slope $-F_{m+1}/F_m$. Thus the positive cone $\mathbb{Z}_+ \oplus \mathbb{Z}_+$ opens into a fan-shaped wedge bordered by these two half-lines. Any integer coordinate point within the wedge comes from an integer coordinate point in the original positive cone. Since $\lim_{m\to\infty} F_{m+1}/F_m = \frac{1+\sqrt{5}}{2}$, the limit cone is just the half-space $\{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \ : \frac{1+\sqrt{5}}{2} a + b \geq 0\}$. Every integer coordinate point in it belongs to some intermediate wedge and so lies in $K_{0+}(\mathcal{A})$. The latter is shown in Fig. 15.

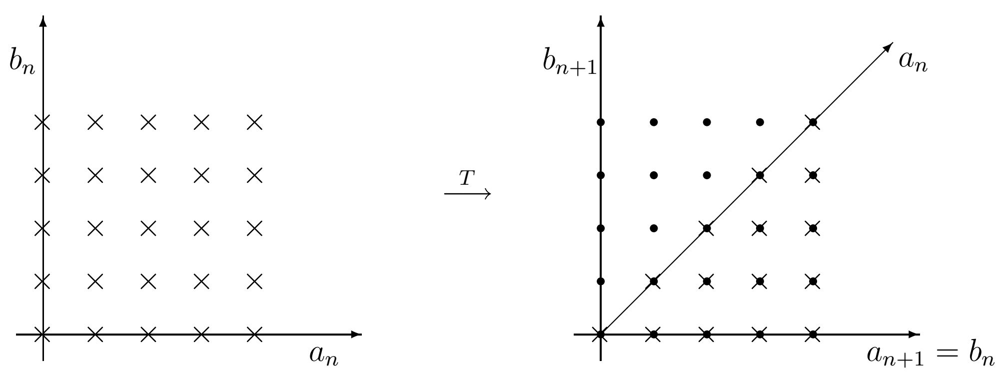


Figure 12: The image of $\mathbb{Z}_+ \oplus \mathbb{Z}_+$ under T.

We shall now evaluate the K-theory of the posets associated with the corresponding Bratteli diagrams in Section 3.2. The strategy will be the same as the one used for the algebra of the Penrose Tiling and will consist essentially of three steps:

1. Construct, as in Fig. 11 the inclusion maps T in (4.23) from the stable part of the Bratteli diagram.

2. Prove that T is invertible over the integer numbers. As a consequence, K_0 will be the direct sum of as many copies of $\mathbb{Z}$ as the number of points k_{n_0} in the stable level n_0 of the corresponding Bratteli diagram of the poset.

3. Identify the subset $T(K_{0+}(\mathcal{A}_{n_0})) \subset K_{0+}(\mathcal{A}_{n_0+1})$ with $K_{0+}(\mathcal{A}_{n_0})$;
   evaluate $T^{-1}(K_{0+}(\mathcal{A}_{n_0}))$;
   get the limit $K_{0+}(\mathcal{A}) = \lim_{m\to\infty} T^{-m} K_{0+}(\mathcal{A}_{n_0})$.

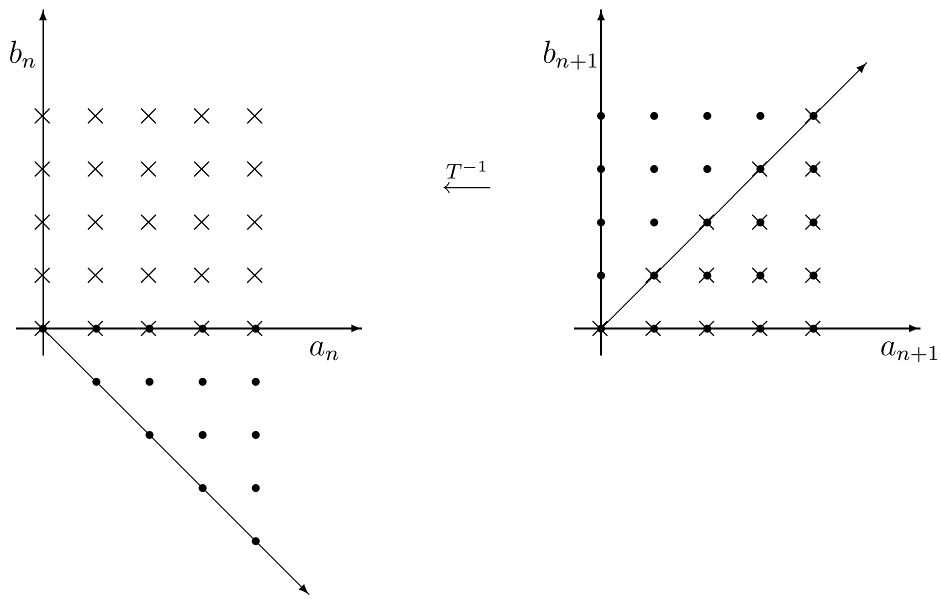


Figure 13: The image of $\mathbb{Z}_+ \oplus \mathbb{Z}_+$ under T^{-1}.

Example 4.1
The K-theory of the poset $\bigvee$. From the stable part of the corresponding Bratteli diagram in Fig. 3, we get for the inclusions (4.23), (4.24) and their inverses the integer valued matrices

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \; . \tag{4.37}$$

Since T is invertible over $\mathbb{Z}$, from definition (4.19), it follows that

$$K_0(\mathcal{A}(\vee)) = \mathbb{Z}^3 \; . \tag{4.38}$$

On the other side, with $(a, b, c) \in \mathbb{Z}^3_+$, one finds that

$$T^{-m} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b - m(a + c) \\ c \end{pmatrix} \; . \tag{4.39}$$

While $a, c \geq 0$, $b - ma - mc$ can become any negative integer provided that $a, c \neq 0$. Therefore

$$K_{0+}(\mathcal{A}(\bigvee)) = \{(a, b, c) \in \mathbb{Z}^3 \;\; | \;\; \begin{array}{lll} a \in \mathbb{Z}_+ \; , c \in \mathbb{Z}_+ & & \\ b \in \mathbb{Z} & if & (a, c) \neq (0, 0) \\ b \in \mathbb{Z}_+ & if & (a, c) = (0, 0) \; \} \; . \end{array} \tag{4.40}$$

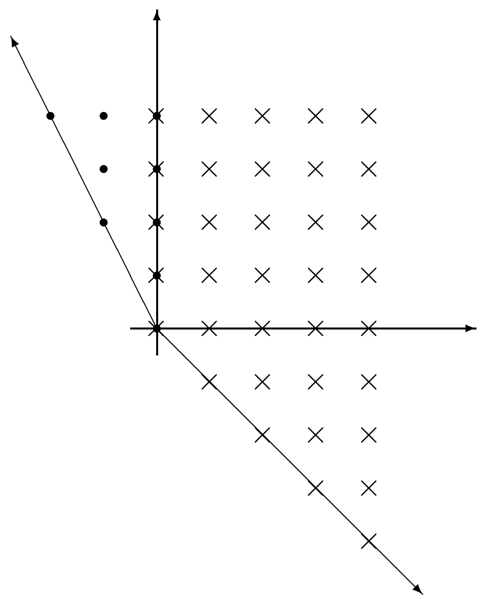


Figure 14: The image of $\mathbb{Z}_+ \oplus \mathbb{Z}_+$ under T^{-2}.

Example 4.2
The K-theory of the poset $|/|$. From the stable part of the corresponding Bratteli diagram in Fig. 7, we get for the inclusions (4.23), (4.24) and their inverses the integer valued matrices

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ . \tag{4.41}$$

Since T is invertible over $\mathbb{Z}$, from definition (4.19), it follows that

$$K_0(\mathcal{A}(|/|)) = \mathbb{Z}^4 \ . \tag{4.42}$$

On the other side, with $(a, b, c, d) \in \mathbb{Z}^4_+$, one finds that

$$T^{-m} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b - m(a + d) \\ c - md \\ d \end{pmatrix} \ . \tag{4.43}$$

As a consequence,

$$K_{0+}(\mathcal{A}(|/|)) = \{(a, b, c, d) \in \mathbb{Z}^4 \ \mid \ \begin{array}{lll} a \in \mathbb{Z}_+ \ , d \in \mathbb{Z}_+ & & \\ b \in \mathbb{Z} \ , c \in \mathbb{Z} & if & a \neq 0 \ , d \neq 0 \\ b \in \mathbb{Z} \ , c \in \mathbb{Z}_+ & if & a \neq 0 \ , d = 0 \\ b \in \mathbb{Z}_+ \ , c \in \mathbb{Z}_+ & if & (a, c) = (0, 0) \ \} \ . \end{array} \tag{4.44}$$

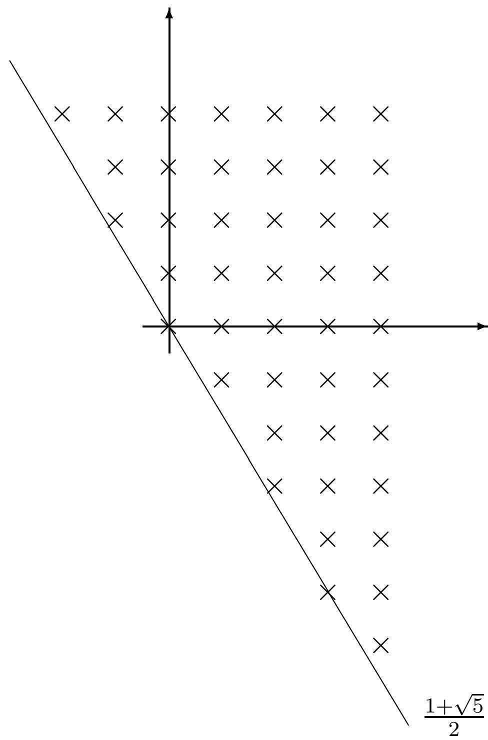


Figure 15: $K_{0+}(\mathcal{A})$ for the Penrose tiling.

Example 4.3
The K-theory of the poset $P_4(S^1)$. From the stable part of the corresponding Bratteli diagram in Fig. 8, we get for the inclusions (4.23), (4.24) and their inverses the integer valued matrices

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.45}$$

Since T is invertible over $\mathbb{Z}$, from definition (4.19), it follows that

$$K_0(\mathcal{A}(P_4(S^1))) = \mathbb{Z}^4 . \tag{4.46}$$

On the other side, with $(a, b, c, d) \in \mathbb{Z}^4_+$, one finds that

$$T^{-m} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b - m(a+d) \\ c - m(a+d) \\ d \end{pmatrix} . \tag{4.47}$$

As a consequence,

$$
\begin{array}{lll}
K_{0+}(\mathcal{A}(P_4(S^1))) = \{(a,b,c,d) \in \mathbb{Z}^4 \ | & a \in \mathbb{Z}_+ \ , d \in \mathbb{Z}_+ & \\
& b \in \mathbb{Z} \ , c \in \mathbb{Z} & if \quad a \neq 0 \quad or \quad d \neq 0 \\
& b \in \mathbb{Z}_+ \ , c \in \mathbb{Z}_+ & if \quad (a,c) = (0,0) \ \} \ .
\end{array}
\tag{4.48}
$$

Notice that as abelian group, $K_0(\mathcal{A}(|\!/\!|)) = K_0(\mathcal{A}(P_4(S^1))) = \mathbb{Z}^4$. But as abelian ordered group $(K_0, K_{0+})\mathcal{A}(|\!/\!|) \neq (K_0, K_{0+})(\mathcal{A}(P_4(S^1)))$, since the positive cone K_{0+} is not the same in the two groups.

Example 4.4
The K-theory of the poset $P_6(S^2)$. From the stable part of the corresponding Bratteli diagram in Fig. 9, we get for the inclusions (4.23), (4.24) and their inverses the integer valued matrices

$$
T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ , \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ . \tag{4.49}
$$

Since T is invertible over $\mathbb{Z}$, from definition (4.19), it follows that

$$
K_0(\mathcal{A}(P_4(S^1))) = \mathbb{Z}^6 \ . \tag{4.50}
$$

On the other side, with $(a,b,c,d,e,f) \in \mathbb{Z}^6_+$, one finds that

$$
T^{-m} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b - m(a+f) \\ c + m^2(a+f) - m(b+e) \\ d + m^2(a+f) - m(b+e) \\ e - m(a+f) \\ f \end{pmatrix} \ . \tag{4.51}
$$

As a consequence,

$$
\begin{array}{llll}
K_{0+}(\mathcal{A}(P_6(S^2))) = \{(a,b,c,d,e,f) \in \mathbb{Z}^6 \ | & & a \in \mathbb{Z}_+ \ , f \in \mathbb{Z}_+ & \\
c \in \mathbb{Z} \ , d \in \mathbb{Z} & & b \in \mathbb{Z} \ , e \in \mathbb{Z} & if \quad a \neq 0 \quad or \quad f \neq 0 \\
& & b \in \mathbb{Z}_+ \ , e \in \mathbb{Z}_+ & if \quad (a,f) = (0,0) \\
c \in \mathbb{Z}_+ \ , d \in \mathbb{Z}_+ & & & if \quad (a,b,e,f) = (0,0,0,0) \ \} \ .
\end{array}
\tag{4.52}
$$

5 Final Remarks

As mentioned before, the construction of the K-theory groups for noncommutative lattices is a preliminary step in the classification and construction of bundles over them and for the theory of characteristic classes. Notably, one would like to construct non trivial bundles, like, for instance, the analogue of the monopole bundle over the lattices approximating the 2-dimensional sphere and non trivial 'topological charges'. Work in this direction is in progress and will be reported elsewhere [16].

Acknowledgments

Much of this work has been done while G.L. and P.T-S were at ESI in Vienna. They would like to thank G. Marmo and P. Michor for the invitation and all people at the Institute for the warm and friendly atmosphere. We are grateful to A.P. Balachandran, G. Bimonte, F. Lizzi e G. Sparano for many fruitful discussions and useful advises. We thank U. Bruzzo and R. Catenacci and J. Varilly for carefully reading the compuscript. J. Varilly deserves a special mentions for suggestions which improved an early version.

We thank the 'Istituto Italiano per gli Studi Filosofici' in Napoli for partial support. The work of P.T-S. was also supported by the Department of Energy, U.S.A. under contract number DE-FG-02-84ER40173. The work of G.L. was partially supported by the Italian 'Ministero dell' Università e della Ricerca Scientifica'.

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